Diffusion through piecewise washboard potential

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We consider the influence of small periodic oscillations of barriers on the stationary motion of a particle through a piecewise washboard potential. Up to the second order in the amplitude of oscillation the corrections to the flux can be both positive and negative and, for equal widths of the well and barrier, they do not depend on the frequency of the oscillations. [S1063-651X(99)05812-2]

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The qualitative characteristics of nonlinear problems have many generic properties which are not too sensitive to a concrete form of nonlinearity. Then, the correct strategy will be—along with numerical and approximate solutions for complicate nonlinear potentials—to choose the simplest form of nonlinearity which allows an analytical solution.

Recently [1] we have presented analytical solutions for one-dimensional diffusion of a classical particle through a bistasble piecewise potential. For the time-dependent problem, our method is highly efficient when one knows the exact solution of the corresponding time-independent problem.

In this Brief Report we consider a particle moving in a washboard potential which is composed from elementary units shown by solid lines in Fig. 1 and extended in both directions to $\pm\infty$ (dotted lines in Fig. 1) in such a way that both ends converge to a common point. A washboard potential represents a sum of constant and periodic force which appears in many physical problems. A typical example is the Josephson junction where the driving force acting on the phase difference between electron pair across the junction consists of constant and sinusoidal forces [2], i.e., the potential has a form $-ax-b\cos(x)$. We show in Fig. 1 this potential for $a=5$ and $b=15$ together with our washboard potential composed of segments of straight lines. Another examples of the problems with the washboard potential are charge density waves [3], phase locking in electric circuits [4], mode locking in ring laser gyroscopes [5], motion of fluxons in superconductors [6], penetration of biological channels by ions [4], and others.

Hence, in this Brief Report we consider a particle moving in a washboard potential shown in Fig. 1. Each barrier has a height V_1 from the right side, and the height V_2 from the left side, and all barriers are subjected to periodic oscillations with frequency Ω. Although the widths of the wells and the barriers have to be different to ensure the correspondence with the Josephson junctions (Fig. 1), we consider the slightly simpler case of equal widths. We assume that the particle moves from one potential minimum to another one due to a Gaussian random force of strength D. When a particle reaches the position of jumps of potential it is exposed to the δ-functional force. The more convenient procedure which we use in what follows is one that takes into account this force by matching the solutions in two regions adjacent to the jump of potential.

The stationary state of such system is described by the constant flux J supported from outside in a downward direction. We find first the solution of the Fokker-Planck equation which corresponds to the constant flux. Then we add a periodic signal $A\cos(\Omega t)$ to the system, which we assume to act on the potential barrier as it is shown in Fig. 1. Our aim is to find the corrections to the flux J caused by the periodic force. Two nonobvious results have been obtained in the second-order perturbation theory in A/D. First, for the case considered of equal widths of the well and the barrier, the corrections to the flux do not depend on the frequency Ω of the field, and, secondly, these corrections may reduce the initial flux when the ratio of the heights of the potential barriers is small enough.

The Fokker-Planck equation for the probability density function $P(x,t)$ for the position x of a diffusive particle at time t is

$$\frac{\partial P}{\partial t}=\frac{\partial}{\partial x}\left[\frac{\partial U}{\partial x}P+D\frac{\partial P}{\partial x}\right]\equiv-\frac{\partial J}{\partial x}, \tag{1}$$

where J is the probability current. For the piecewise potentials, $\partial U/\partial x=0$, and Eq. (1) reduces to a simple diffusion equation. However, the barrier heights enter the matching conditions, namely, one has to solve Eq. (1) in each region of $U(x)=\text{const}$, and then to ensure the continuity of J across the boundary of these regions. The matching conditions at points z of a finite jump of the potential $U(x)$ have the following form [7]:

$$P(z+0,t)\exp\left(\frac{U(z+0)}{D}\right)=P(z-0,t)\exp\left(\frac{U(z-0)}{D}\right), \tag{2}$$

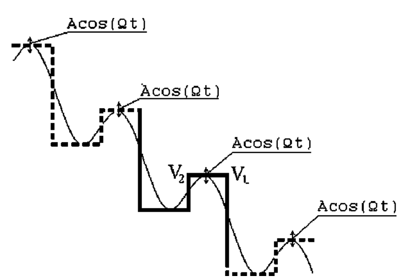

FIG. 1. Washboard piecewise potential with oscillating barriers. Here V_1 and V_2 are the barrier's heights with respect to left and right wells, respectively. All the barriers are subjected to oscillations of the form $A\cos(\Omega t)$. The thin lines represent the Josephson potential $-5x-15\cos(x)$.

$$\frac{\partial P(z+0,t)}{\partial x}=\frac{\partial P(z-0,t)}{\partial x}. \tag{3}$$

Denoting the probability function in the region $-1<x<0^-$ as $P_1(x,t)$ and that in region $0^+<x<1$ as $P_2(x,t)$ one can rewrite the matching conditions (2) and (3) as

$$P_1(0,t)=P_2(0,t)\exp\left(\frac{V_1+A\cos(\Omega t)}{D}\right), \tag{4}$$

$$\frac{\partial P_1(0,t)}{\partial x}=\frac{\partial P_2(0,t)}{\partial x}. \tag{5}$$

The matching condition at $x=1$ are similar to Eq. (5) if one takes into account that due to the periodic repetition of the elementary blocks the distribution function at the point $x=1^+$ is equal to that at $x=-1^+$, i.e.,

$$P_2(1,t)\exp\left(\frac{V_2+A\cos(\Omega t)}{D}\right)=P_1(-1,t). \tag{6}$$

Finally, the flux entering the elementary block at $x=-1$ has to be equal to the flux flowing out of this block at $x=1$,

$$\frac{\partial P_1(x=-1,t)}{\partial x}=\frac{\partial P_2(x=1,t)}{\partial x}. \tag{7}$$

The expansion of $\exp(A\cos(\Omega t)/D)$ in a series of modified Bessel function of the first kind [8] yields

$$\exp\left(\frac{A\cos(\Omega t)}{D}\right)=\sum_{l=-\infty}^{\infty} I_l\left(\frac{A}{D}\right)\cos(l\Omega t). \tag{8}$$

Using expansion (8) in Eqs. (4) and (6) one seeks a solution of Eq. (1) in regions $m=1,2$ in the form

$$P_m=S_m+\sum_{l=1}^{\infty}\left(\frac{A}{D}\right)^l f_m^l(x,t). \tag{9}$$

We keep only two corrections of lowest order in A/D to the field-free probabilities S_m. Then, one has to keep the following terms in Eq. (9):

$$P_m=S_m+\frac{A}{D}\{[f_m\exp(rx)+\tilde{f}_m\exp(-rx)]\exp(i\Omega t)+\text{c.c.}\}+\left(\frac{A}{D}\right)^2(g_m+z_m x), \tag{10}$$

where $r=(i\Omega/D)^{1/2}$ and c.c. stands for the complex conjugate.

For the time-independent part of the solutions one immediately finds from Eqs. (4) and (6) that

$$S_1=\left[\psi\left(\frac{V_1}{D},\frac{V_2}{D}\right)\exp\left(\frac{V_1}{D}\right)-x\right]\frac{J}{D}, \tag{11}$$

$$S_2=\left[\psi\left(\frac{V_1}{D},\frac{V_2}{D}\right)-x\right]\frac{J}{D}, \tag{12}$$

where

$$\psi\left(\frac{V_1}{D},\frac{V_2}{D}\right)=\frac{1+\exp\left(\frac{V_2}{D}\right)}{\exp\left(\frac{V_2}{D}\right)-\exp\left(\frac{V_1}{D}\right)}. \tag{13}$$

Collecting now all terms of the first order in A/D in Eqs. (4), (5), (6), and (7) one obtains, respectively,

$$f_1+\tilde{f}_1=\left(\frac{S_2(x=0)}{2}+f_2+\tilde{f}_2\right)\exp\left(\frac{V_1}{D}\right), \tag{14}$$

$$f_1-\tilde{f}_1=f_2-\tilde{f}_2, \tag{15}$$

$$\left[\frac{S_2(x=1)}{2}+f_2\exp(r)+\tilde{f}_2\exp(-r)\right]\exp\left(\frac{V_2}{D}\right)=[f_1\exp(-r)+\tilde{f}_1\exp(r)], \tag{16}$$

$$f_2\exp(r)-\tilde{f}_2\exp(-r)=f_1\exp(-r)-\tilde{f}_1\exp(r). \tag{17}$$

It follows from Eqs. (15) and (17) that $f_1=-\tilde{f}_2$ and $f_2=-\tilde{f}_1$. The latter together with Eqs. (14) and (16), leads to

$$f_1(r)=\tilde{f}_1(-r)=\frac{1}{2[\exp(-r)-\exp(r)]}\times\left[\frac{S_2(x=1)}{[1+\exp(-V_2/D)]}-\exp(r)\frac{S_2(x=0)}{[1+\exp(-V_2/D)]}\right]. \tag{18}$$

Repeating the same procedure for the second order in (A/D) terms in the distribution function (10) and using the normalization condition one obtains

$$g_1=-g_2=\frac{[S_2(x=0)+2(f_2+\tilde{f}_2+\text{c.c.})]}{4[1+\exp(-V_1/D)]}, \tag{19}$$

$$z_1=z_2=g_1-\frac{1}{4[1+\exp(-V_2/D)]}\times[S_2(x=1)+2(f_2\exp(r)+\tilde{f}_2\exp(-r)+\text{c.c.})]. \tag{20}$$

Our prime interest is the change of the flux due to the oscillations of barriers. The renormalized flux $\tilde{J}$ is defined according to Eq. (10) as $J+(A/D)^2 z_1$ which with the use of the foregoing formulas takes the following simple form:

$$\tilde{J}=J\left[1+\left(\frac{A}{D}\right)^2\times\frac{1-v_1-v_2-6v_1v_2-v_1v_2(v_1+v_2)+v_1^2v_2^2}{4(1+v_1)^2(1+v_2)^2}\right],$$

$$v_{1,2} \equiv \exp\left(\frac{V_{1.2}}{D}\right). \quad (21)$$

Two interesting conclusions follow from Eq. (21):

(1) The renormalized flux is independent of the frequency Ω of oscillations. The latter which enters all the foregoing formulas through $r=(i\Omega/D)^{1/2}$ does not appear in Eq. (21). However, the dependence on frequency will appear if the width of the well is different from the width of the barrier as in the case of Josephson junctions.

(2) The correction to the flux induced by oscillations can be both positive and negative depending on the barrier heights. For example, for $v_1 \equiv \exp(V_1/D)=10$, the correction to the flux is positive for $v_2 \equiv \exp(V_2/D)>2$, and negative for $v_2 \equiv \exp(V_2/D)<2$.

The toy example of our system is a waterfall of constant power when the water is falling through cascade of horizontal steps with notches (barriers) along the end of each step. The question is what is the influence of the small periodic oscillations of the barriers (or steps) on the power of the waterfall. The answer is that such oscillations may both increase or decrease the power, and this influence is independent on the frequency of oscillations.

The graph of Josephson potential in Fig. 1 shows a clear similarity between our model and the real problems involving washboard potentials. However, the qualitative comparison cannot be performed since we consider a motion in a periodically changing flat potential while in real problems the periodic force acts on a moving particle. This difference results, in particular, in the correction ΔJ to a flux [Eq. (21)] which does not depend on frequency for the given geometry, in contrast to the strong frequency dependence for the Josephson junctions (''Shapiro'' steps).

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